1.) Suppose $g(x)$ is a continuously differentiable function with $g(1) = 2$ and $g'(1) = 3$. Let $h(x) = (g(x))^2$.

a.) Find $h'(1)$.

b.) Find the equation of the tangent line to the graph $y = h(x)$ at $x = 1$.

c.) Evaluate the limit

$$\lim_{x\to 1} \frac{\sqrt{h(x)} - 2}{g(x) - 2}.$$

Solution:

a.) $h'(x) = 2g(x)g'(x)$, so $h'(1) = 2 \cdot 2 \cdot 3 = 12$

b.) $h(1) = 4$ so $y = 12x - 8$

c.) $\lim_{x\to 1} \frac{\sqrt{h(x)}-2}{g(x)-2}$ is of indeterminate form $\frac{0}{0}$ so apply l'Hospital's rule: $\lim\limits_{x\to 1} \dfrac{\sqrt{h(x)} - 2}{g(x) - 2} = \lim\limits_{x\to 1} \dfrac{\frac{1}{2}(h(x))^{-1/2}h'(x)}{g'(x)} = \dfrac{\frac{1}{2} \cdot \frac{1}{2} \cdot 12}{3} = 1$

Scoring Rubric:

$2 : \begin{cases} 1: & h'(x) \\ 1: & \text{answer} \end{cases}$

$3 : \begin{cases} 1: & h(1) = 4 \\ 2: & \text{equation of line} \end{cases}$

$4 : \begin{cases} 2: & \text{l'Hospital's Rule} \\ 2: & \text{answer} \end{cases}$

2.) Let $F(x) = \int_1^x f(t)\,dt$, where $f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du$.

a.) Over the interval $1 \le x \le 10$ is $F(x)$ concave up or down? Explain your reasoning.

b.) Find $F''(2)$.

c.) Find $\lim_{x\to 1} \frac{F(x)}{x^2 - 2x + 1}$.

Solution:

a.) $F'(x) = \int_1^{x^2} \frac{\sqrt{1+u^4}}{u}\,du$

$F''(x) = \frac{\sqrt{1+(x^2)^4}}{x^2} \cdot \frac{d(x^2)}{dx} = \frac{2\sqrt{1+x^8}}{x}$

$F''(x) > 0$ for $1 \le x \le 10$

$F(x)$ is concave up

b.) $F''(2) = \frac{2\sqrt{1+2^8}}{2} = \sqrt{257}$

c.) $F(1) = 0$ and $1^2 - 2(1) + 1 = 0$ so $\lim_{x\to 1} \frac{F(x)}{x^2-2x+1}$ is of indeterminate form $\frac{0}{0}$. Apply l'Hospital's Rule:

$\lim_{x\to 1} \frac{F(x)}{x^2-2x+1} = \lim_{x\to 1} \frac{F'(x)}{2x-2}$

$F'(1) = 0$ and $2(1) - 2 = 0$ so again we have indeterminate form $\frac{0}{0}$. Apply l'Hosptial's Rule:

$\lim_{x\to 1} \frac{F'(x)}{2x-2} = \lim_{x\to 1} \frac{F''(x)}{2} = \frac{\sqrt{1+1^8}}{1} = \sqrt{2}$.

Scoring Rubric:

$4 : \begin{cases} 1: & F'(x) \\ 1: & F''(x) \\ 2: & F''(x) > 0 \text{ and answer} \end{cases}$

$1 : \{ 1: \text{ answer}$

$4 : \begin{cases} 2: & \text{first l'Hospital's correctly} \\ 2: & \text{2nd l'Hospital's correctly} \end{cases}$

3.) A rectangular box with square top and bottom is to be made of two materials. The material for the top and bottom costs \$5 per square foot and the material for the sides costs \$3 per square foot.

a.) Suppose the total volume is to be 45 cubic feet. Find the dimensions that minimize the cost.

b.) Suppose the total cost is set to \$120. Find the dimensions that maximize the volume.

Solution:

a.) Let x be the side length of the square top and bottom and let y be the height of the box. The combined area of top and bottom is $2x^2$ and the area of the sides of the box is $4xy$, so the total cost of the box is

$$C(x, y) = 10x^2 + 12xy.$$

The volume is $x^2y = 45$, so we can eliminate y from $C(x, y)$ to obtain

$$C(x) = 10x^2 + 12x \cdot \frac{45}{x^2} = 10x^2 + \frac{12 \cdot 45}{x}.$$

$$C'(x) = 20x - \frac{12 \cdot 45}{x^2} = 0$$

means that $x = 3$, and then from $x^2y = 45$ we find $y = 5$. So the dimensions that minimize cost are $x = 3$ feet and $y = 5$ feet.

b.) $120 = 10x^2 + 12xy$ and $V(x, y) = x^2y$

Eliminate y from $V(x, y)$

$V(x, y) = x^2 \left(\frac{120-10x^2}{12x}\right) = 12x - \frac{5}{6}x^3.$

$V'(x) = 10 - \frac{5}{2}x^2 = 0$ means $x = 2$. Thus $y = \frac{10}{3}$. The dimensions that maximize the volume are $x = 2$ feet and $y = 10/3$ feet.

Scoring Rubric:

$$5 : \begin{cases} 1 : \text{ cost function} \\ 1 : \text{ volume function} \\ 1 : \text{ putting cost in terms of } x \\ 1 : \text{ derivative of cost} \\ 1 : \text{ answer} \end{cases}$$

$$4 : \begin{cases} 1 : \text{ cost and volume} \\ 1 : \text{ putting volume in terms of } x \\ 1 : \text{ derivative of volume} \\ 1 : \text{ answer} \end{cases}$$