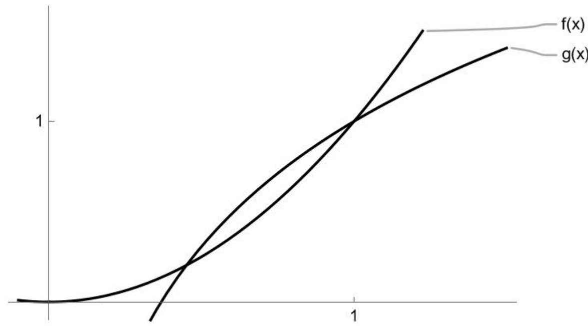
1.) Consider the region R between the curves $f(x) = x^2$ and $g(x) = 1 + \ln x$ over the interval $[a, 1]$ where $0 < a < 1$ and $f(a) = g(a)$.



a.) Find the area of the region enclosed by the graphs of f and g.

b.) Find the volume of the washer generated when R is revolved around the x-axis.

c.) Find the volume of the washer generated when R is revolved around the y-axis.

Solution:

a.) $x^2 = 1 + \ln x \Rightarrow x = 1,\ x = a = 0.450764$

$\int_a^1 (g(x) - f(x))\ dx = 0.056371$

The area of the region is 0.056.

b.) $\int_a^1 \pi \left(g(x)^2 - f(x)^2\right)\ dx = 0.209746$

The volume is 0.022.

c.) $f(a) = g(a) = 0.203188 = b$

$f(x) = x^2 \Rightarrow f(y) = \sqrt{y}$

$g(x) = 1 + \ln x \Rightarrow g(y) = e^{y-1}$

$\int_b^1 \pi \left(f(y)^2 - g(y)^2\right)\ dy = 0.254316$

$$\left(\begin{array}{l}\text{Alternatively:}\\ \int_a^1 2\pi x\,(g(x) - f(x))\ dx = 0.254316\end{array}\right)$$

The volume is 0.254.

Scoring Rubric:

$$3 : \begin{cases} 1 : \text{ bounds of integration} \\ 1 : \text{ integrand} \\ 1 : \text{ answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{ integrand} \\ 1 : \text{ answer} \end{cases}$$

$$4 : \begin{cases} 1 : \text{ rewrite } f(x) \text{ and } g(x) \text{ in terms of } y \\ \quad\ \ \text{OR set up using cylinders} \\ 1 : \text{ bounds of integration} \\ 1 : \text{ integrand} \\ 1 : \text{ answer} \end{cases}$$

2.) Harriet the snail is moving along a straight, skinny rail. Her velocity at time t is $v_H(t) = \sqrt[3]{\cos(t)}$ for $0 \le t \le 3$. At time $t = 0$ Harriet is at position 2 along the rail.

Thomas the snail is moving along the same straight, skinny rail. His velocity at time t is $v_T(t) = (t-1)(0.8^t)$ for $0 \le t \le 3$. At time $t = 0$ Thomas is at position 3 along the rail.

a.) Find Harriet's and Thomas's positions along the rail at time $t = 2$.

b.) Are Harriet and Thomas racing towards each other or away from each other at time $t = 2$? Explain your reasoning.

c.) Find the total distance travelled by Harriet over the time interval $0 \le t \le 3$.

Solution:

a.) $x_H(2) = 2 + \int_0^2 v_H(t)\ dt = 3.051724$

At time $t = 2$, Harriet is at 3.052.

$x_T(2) = 3 + \int_0^2 v_T(t)\ dt = 2.880396$

At time $t = 2$, Thomas is at 2.880.

b.) $v_H(2) = \sqrt[3]{\cos(2)} = -0.74659 < 0$

At time $t = 2$, Harriet is moving to the left.

$v_T(2) = (2-1)(.8^2) = .64 > 0$

At time $t = 2$, Thomas is moving to the right.

At time $t = 2$, $x_H(2) > x_T(2)$, so Thomas is to the left of Harriet.

Thus, at time $t = 2$, Harriet and Thomas are on a collision course!!

c.) $\int_0^3 |v_H(t)|\ dt = \int_0^{\pi/2} \sqrt[3]{\cos(t)}\ dt - \int_{\pi/2}^3 \sqrt[3]{\cos(t)}\ dt = 1.293555 - (-1.152121) = 2.445676$

Over the time interval $0 \le t \le 3$, the total distance traveled by Harriet is 2.446.

Scoring Rubric:

a.) $3 : \begin{cases} 1: & \text{definite integrals} \\ 1: & \text{Harriet's position} \\ 2: & \text{Thomas's position} \end{cases}$

b.) $3 : \begin{cases} 1: & \text{direction of motion} \\ 2: & \text{answer with explanation} \end{cases}$

c.) $3 : \begin{cases} 1: & \text{definite integral} \\ 2: & \text{answer} \end{cases}$

t (hours)	1	3	6	8	9
$m(t)$ (square feet)	151.02	72.97	18.97	7.19	4.39

3.) At a daycare facility the amount of floor space which is still uncluttered with toys t hours after the facility opens for the day is given by a differentiable function m, where $m(t)$ is measured in square feet. Selected values of $m(t)$ are given in the table above.

a.) Use the data in the table to estimate $m'(7)$. Using correct units, interpret the meaning of $m'(7)$ in the context of the problem.

b.) Explain why there must be at least one time t, for $1 < t < 9$, such that $m'(t) = -18$.

c.) The clear space on the floor, in square feet, can also be modeled by a function C, given by $C(x) = \frac{1}{1000}(-1.3x^4 + 83x^3 - 1476x^2 - 870x + 158005)$, where x is the number of children at the facility. When there are 21 square feet of clear floor space the number of children is increasing at a rate of 0.2 children per hour. According to this model, what is the rate of change of amount of floor space with respect to time, in square feet per hour, at the time when there are 21 square feet of clear floor space?

Solution:

a.) $m'(7) \approx \frac{m(8)-m(6)}{8-6} = \frac{7.19-18.97}{2} = -5.89$

The amount of clear floor space is changing at approximately -5.89 square feet per hour at time $t = 7$.

b.) $\frac{m(6)-m(3)}{6-3} = \frac{18.97-72.97}{6-3} = -18$

By the Mean Value Theorem, there exists a value c, $3 < c < 6$, such that $m'(c) = -18$.

c.) $C(x) = 21 \Rightarrow x = 16$

$\frac{d}{dt}(C(x)) = \frac{(-5.2x^3+249x^2-2952x-870)}{1000} \cdot \frac{dx}{dt}$

$\frac{d}{dt}(C(x))\Big|_{x=5} = \frac{(-5.2(16)^3+249(16)^2-2952(16)-870)}{1000} \cdot$
$0.25 = -1.4143$

The rate of change of the clear floor space with respect to time is -1.4143 square feet per hour.

Scoring Rubric:

a.) $3 : \begin{cases} 2: \text{ estimate} \\ 1: \text{ interpretation with units} \end{cases}$

b.) $2 : \begin{cases} 1: \frac{W(6)-W(3)}{6-3} \\ 1: \text{ conclusion using} \\ \quad\;\text{Mean Value Theorem} \end{cases}$

c.) $4 : \begin{cases} 1: \text{ find } x \\ 2: \text{ derivative} \\ 1: \text{ answer} \end{cases}$