## Activators

## Calculus Field Day 2023

## Car convoys

On a very long, single-lane, one-way road, a hundred cars are traveling. Initially no two cars go at the same speed, but whenever a faster car gets stuck behind a slower car, it simply slows down to the speed of the car in front of it and stays close behind that car. There is no way to pass. Gradually, convoys are formed of cars that travel together. These convoys can grow and even merge. After a long time, the convoys stop evolving. The questions below are about these final convoys.

1. If the cars end up traveling in two convoys, which convoy is the faster one, the one in front or the one in the back?
2. What if there are more than two convoys? is the fastest convoy in the front, in the back, or could it be any of the convoys? Can you say something about the relative speeds of all the convoys?
3. On average, how many cars are in the last convoy (the one that's all the way in the back)? What can you say about the lead car of this last convoy?
4. Do the convoys tend to get longer going from the front convoy to the back convoy? shorter? or are they all more or less the same length?

## Car convoys (solution)

As long as there is a fast car (or fast convoy) trailing behind a slow car (or convoy), the fast convoy will eventually catch up to the slow convoy, slow down and merge with the slow convoy. Therefore, once the convoys stabilize, there cannot be a fast convoy behind a slow convoy. Therefore the leading convoy is the fastest, the one behind it is slower, the one behind that slower still, all the way to the slowest convoy in the back.

The leading car in the last convoy (call it C) must be the slowest car. If there was a slower car ahead of $C$, the last convoy would merge with it eventually. If there was a slower car behind $C$, its convoy would not be able to catch up with C so the convoy led by C would not be the last one. Since the location of the slowest car is random, the last convoy would include half the cars, on average.

The last convoy never interacts with the convoys ahead of it. So we can remove it and get a smaller example of the same question. Therefore, since the last convoy tends to include roughly half the cars, the convoy directly ahead of it will include roughly half of the remaining cars, etc. So the convoys shrink exponentially in size going from back to front.

## Party Hats



The children at Blanca's birthday party are sitting at a round table. Each child gets a party hat that is either red (R), green (G) or blue (B). Children sitting next to each must wear different color hats.

If there are 3 children then there are only six hat color combinations: $R G B, R B G, G R B, G B R$, $R R G$ and BGR.

1. How many hat color combinations are there for 4 children?
2. How many hat color combinations are there for 5 children?
3. Describe a method for finding the combination number for more than 5 children.

## Party Hats (solution)

1. For four children the number of combinations is 18:

RGRG, RGRB, RGBG, RBRG, RBRB, RBGB,
GBGB, GBGR, GBRB, GRGB, GRGR, GRBR,
$B R B R, B R B G, B R G R, B G B R, B G B G, B G R G$.
2, 3. In general, the recursion formula for the number of combinations is $f(n+2)=f(n+1)+2 f(n)$.
One way to see that is to break the choices for $n+2$ into two groups:
Group 1: For children $1, \ldots, n+1$ choose an arbitrary sequence of colors from the $n+1$ case. Since color \#1 and color \#(n+1) are different, there is exactly one color choice left for \#( $n+2$ ). So the total number of choices for this group is $f(n+1)$.

Group 2: For children $1, \ldots, \mathrm{n}$ choose an arbitrary sequence of colors from the n case. For color \#( $\mathrm{n}+1$ ) use the same color \#1. There are now exactly two color choices left for \#(n+2). So the total number of choices for this group is $2 f(n)$.

Group 1 and 2 are disjoint because colors \#1 and \#(n+1) are different in Group 1 and the same in Group 2.
The two groups cover all the choices because for any valid $n+2$ choice, it is easy to see that if colors \#1 and \#( $n+1$ ) are different then the choice comes from Group 1 and otherwise it comes from Group 2.

In particular $f(4)=f(3)+2 f(2)=6+2^{*} 6=18$ as we have seen, and $f(5)=f(4)+2 f(3)=18+2^{*} 6=30$

## Moving Game Pieces (bonus question)

On a game board with 7 positions there are 3 solid tokens and three hollow tokens. The middle position is empty.


At every turn you are allowed to move one token. The rules are:

- A token can be moved to a neighboring position if it is empty
- A solid token can jump over one hollow token, and a hollow token can jump over one solid token, to get to an empty position on the other side.

Your goal is to flip the positions of the solid and hollow tokens.
Challenge: can you solve it with 5 tokens on each side? what about N tokens?

## Moving Game Pieces (solution)

The initial position (with $N=5$ ) is $\mathrm{XXXXX}-00000$. Move the rightmost $X$ one position to the right (call this move X 1 ), to get $\mathrm{XXXX}-\mathrm{X00000}$. Now jump the leftmost 0 to the left, over the right most $X$ (call this move O1). Get XXXX0X-0000. Next shift the rightmost XOX pattern one place to the right (this requires 3 consecutive turns. Call this move X3) to get XXX-X0X0000. Now jump the two leftmost 0s to the left, over the two rightmost Xs (call this move 02). Get XXX0X0X-000. Continue in a similar fashion with an $X 5$ move, an 03 move, an $X 7$ move, then 04 and $X 9$. You will get -X0X0X0X0X0. Now take the next step in the sequence, 05 . You get OXOXOXOXOX-. This sequence is exactly the same as the previous sequence, if you turn it 180 degrees. So you can retrace your steps on the inverted sequence and end up at the inversion of the initial position, 00000-XXXXX.

## Weights (bonus question)

A merchant of gemstones has a balance scale. She uses one plate for the gems and the other plate for the weights. She wants to buy enough different weights in order to be able to weigh all possible customer orders of 1 gram, 2 grams, etc. - all the way up to 31 grams of gems. She wants to buy the smallest possible number of weights. Which weights should she buy?


Her friend points out that if she places weights on both plates of the scale, she can measure orders up to 40 grams with the a smaller number of weights - but she would have to change the size of the weights she gets. Which weights should she buy?


## Weights (solution)

If the gems and weights are not mixed, then there are $2^{\wedge} \wedge-1$ non-empty combinations of weights that can be placed on the other plate, so that is the maximum number of product weights that can be measured. To measure 1 g to 31 g , you need five weights of size $1,2,4,8$ and 16.
If the weights can be intermingled with the gems, then there are 3 placement choices per weight (left plate, right plate or absent), giving $3 \wedge N-1$ non-empty choices. Each of these choices can be paired with a symmetrical choice with flipped plates, and the gems must always be placed on the lighter plate, therefore these pairs of choices are equivalent. This leaves a maximum of $(3 \wedge N-1) / 2$ different gem weights. Using the four weights of $1,3,9$ and 27 you can measure weights from 1 g to 40 g .

