## SOLUTIONS:

(a) [2 points: 1 for correct integral setup, 1 point for evaluation]
$\int_{1}^{4} 7 t^{3}-145 t^{2}+772 t d t=3191.25$. Rounding to the nearest whole number, 3191 riders entered the Red Line from 6am to 9am.
(b) [2 points: 1 for correct integral setup, 1 point for evaluation]
$\frac{1}{4-1} \int_{1}^{4} 7 t^{3}-155.5 t^{2}+922.25 t-423.125 d t=942.75$ The average number of riders leaving the Red Line from 6am to 9am is 942.75 riders per hour.
(c) [3 points: 1 for correct quantity being integrated, 1 point for setup of integral, 1 point for evaluation.]
$\int_{0}^{2} H(t)-L(t) d t=\int_{0}^{2}\left(7 t^{3}-145 t^{2}+772 t\right)-\left(7 t^{3}-155.5 t^{2}+922.25 t-423.125\right) d t=$ 573.75 Rounding to the nearest whole number, there are 574 passengers on the Red Line at 7am.
(d) [3 points: 1 for correct interpretation of rate of change of riders, 1 point for evaluation of quantity, 1 point for interpretation + justification]
The rate of change in the number of riders is $H(t)-L(t)$. Since $H^{\prime}(3)-L^{\prime}(3)=$ $-87.25<0$, the rate of change in the number of riders is decreasing at 8am.

## SOLUTIONS:

(a) [3 points: 1 for correct tangent vector, 2 for correct equation (1 for slope)]

Position is $(1,2)$ and tangent vector is the velocity at time $t=0$, which is $(2 \sin (\pi / 3), \sqrt{4})=$ $(\sqrt{3}, 2)$. The slope of the tangent line is $d y / d x=(d y / d t) /(d x / d t)=2 / \sqrt{3}$ so the equation is $y-2=2 / \sqrt{3}(x-1)$ or equivalently $\sqrt{3}(y-2)=2(x-1)$.
(b) [2 points: 1 for correct quantity, 1 point for correct evaluations]

Acceleration is the derivative of velocity, so acceleration vector is $\left(2 \cos \left(7 t^{2}+\pi / 3\right)\right.$. $7,(1 / 2)\left(t^{3}+4\right)^{-1 / 2} \cdot\left(3 t^{2}\right)$, which evaluates to $(14 \cos (\pi / 3), 0)=(7,0)$.
(c) [3 points: 1 point for correct $x$-displacement, 1 point for correct $y$-displacement, 1 point for correct final position]
Displacement is obtained by integrating velocity. The $x$-displacement is $x(2)-x(0)=$ $\int_{0}^{2} 2 \sin (7 t+\pi / 3) d t=0.368$ and so $x(5)=1.368$. Likewise, the $y$-displacement is $y(2)-y(0)=\int_{0}^{2} \sqrt{t^{3}+4} d t=4.821$ and so $y(2)=6.821$. So the position is $(x(2), y(2)=(1.368,6.821)$.
(d) [2 points: 1 point for correct integrand, 1 point for evaluation] Distance traveled is obtained by integrating speed, which is the magnitude of the velocity vector. Thus, the distance is
$\int_{0}^{2} \sqrt{\left[(2 \sin (7 t+\pi / 3)]^{2}+\left[\sqrt{t^{3}+4}\right)\right]^{2}} d t=5.624$. Give zero points for simply finding the distance between the start and end calculated in (c).

## SOLUTIONS:

(a) [2 points: 1 for correct integrand, 1 point for answer]

The area enclosed by a polar curve $r=f(\theta)$ for $a \leq \theta \leq b$ is $\int_{a}^{b} \frac{1}{2} r^{2} d \theta$. Thus, the desired area is $\int_{0}^{2 \pi} \frac{1}{2}[2+\sin (\theta)+\sin (2 \theta)]^{2} d \theta=5 \pi=15.708$.
(b) [2 points: 1 for correct integrand, 1 point for answer]

The arclength of a polar curve $r=f(\theta)$ for $a \leq \theta \leq b$ is $\int_{a}^{b} \sqrt{r^{2}+(d r / d \theta)^{2}} d \theta$.
Thus, the desired arclength is $\int_{0}^{2 \pi} \sqrt{[2+\sin (\theta)+\sin (2 \theta)]^{2}+[\cos (\theta)+2 \cos (2 \theta)]^{2}} d \theta=$ 34.152.
(c) [3 points: 1 point for condition $d r / d \theta=0,1$ point for $d r / d \theta, 1$ point for answer]

The distance to the origin is $r$, so the maximum distance occurs at a point where $d r / d \theta=0$. This yields $\cos (\theta)+2 \cos (2 \theta)=0$ which yields the quadratic equation $4 \cos ^{2} \theta+\cos \theta-2=0$ with solutions $\cos \theta=\frac{-1 \pm \sqrt{33}}{8}$ so that $\theta=0.936,2.574,3.709,5.347$ radians. (Alternatively, one could graph the function of $\theta$ and solve it numerically.) The actual maximum distance, per the graph, occurs in the first quadrant, corresponding to $\theta=0.936$.
(d) [3 points: 1 point for condition $d x / d \theta=0,1$ point for $d x / d \theta, 1$ point for answer] Since $x=r(\theta) \cos \theta$ and $y=r(\theta) \sin \theta$, the slope of the tangent line is $\frac{d y}{d x}=\frac{d y / d \theta}{d x / d \theta}$, so the tangent line is vertical when $d x / d \theta=0$. We obtain $\frac{d x}{d \theta}=\frac{d}{d \theta}[(2+\sin (\theta)+\sin (2 \theta)) \cos \theta]=$ $[\cos (\theta)+2 \cos (2 \theta)] \cos \theta-[2+\sin (\theta)+\sin (2 \theta)] \sin (\theta)$. Graphing this function yields four zeroes: $\theta=0.515,3.440,5.050,5.250$, with the second one corresponding to the angle in the third quadrant: thus $\theta=3.440$.

## SOLUTIONS:

(a) [3 points: 1 point for $h^{\prime}(t), 1$ point for $h^{\prime}(\pi)$ with units, 1 point for interpretation] We have $h^{\prime}(t)=8 \sin (t) \cos (t)+4 \cos (t)$ inches $/ \mathrm{min}$ so $h^{\prime}(\pi)=-4$ inches $/ \mathrm{min}$. This quantity represents the velocity of the boat at time $t=\pi$ minutes, and indicates that the boat is moving downward at 4 inches per minute at time $t=\pi$.
(b) [4 points: 1 for correct integral setup, 2 points for correct symbolic evaluation, 1 point for answer]
The average height is $\frac{1}{2 \pi} \int_{0}^{2 \pi} h(t) d t=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left[4 \sin (t)^{2}+4 \sin (t)-3\right] d t=\frac{1}{2 \pi} \int_{0}^{2 \pi}[2 \cos (2 t)+$ $4 \sin (t)-1] d t=\left.[\sin (2 t)-\cos (t)-t]\right|_{t=0} ^{2 \pi}=-1$ inch.
(c) [2 points: 1 point for interpretation, 1 point for answer]

The boat is above the surface of the dock when $h(t)>0$, which requires $4 \sin ^{2} t+$ $4 \sin (t)-3>0$. Factoring yields $(2 \sin t-1)(2 \sin t+3)>0$ so since the second term is always positive, we have $\sin t>1 / 2$ so that $\pi / 3<t<2 \pi / 3$. Therefore, the total amount of time is $\pi / 3$ minutes.
(d) [1 point]

The average height is $\frac{1}{\pi / 3} \int_{\pi / 3}^{2 \pi / 3} h(t) d t=\frac{1}{\pi / 3} \int_{\pi / 3}^{2 \pi / 3}\left[4 \sin (t)^{2}+4 \sin (t)-3\right] d t$.

## SOLUTIONS:

(a) [3 points: 1 for correct function, 1 for correct limits, 1 for value]

The area is $\int_{1}^{9}(\sqrt{x}-1) d x=\frac{2}{3} x^{3 / 2}-\left.x\right|_{x=1} ^{9}=28 / 3$.
(b) [3 points: 1 for correct cross-sectional area, 1 for correct integral setup, 1 for answer] The cross-sections are annular regions with outer radius $\sqrt{x}$ and inner radius 1 , so the area of each cross-section is $\pi[\sqrt{x}]^{2}-\pi 1^{2}=\pi(x-1)$. Therefore, the volume is $\int_{1}^{9} \pi(x-1) d x=\frac{1}{2} x^{2}-\left.x\right|_{x=1} ^{9}=32 \pi$.
(c) [4 points: 1 for correct cross-sectional area, 1 for correct integral setup, 1 for symbolic evaluation, 1 for answer]
The cross-sections are squares regions with side length $\sqrt{x}-1$, so the area of each cross-section is $(\sqrt{x}-1)^{2}=x-2 \sqrt{x}+1$. Therefore, the volume is $\int_{1}^{9}(x-2 \sqrt{x}+1) d x=$ $\frac{1}{2} x^{2}-\frac{4}{3} x^{3 / 2}+\left.x\right|_{x=1} ^{9}=40-4 / 3(26)+8=40 / 3$.

## SOLUTIONS:

(a) [3 points: 1 for using ratio or root test, 1 for correct interval endpoints, 1 for correctly excluding both endpoints]
Using the ratio test, with $a_{n}=(-1)^{n+1} \frac{n}{4^{n}} x^{n}$ we have $\left|a_{n+1} / a_{n}\right|=\frac{n+1}{4 n}|x|$ whose limit as $n \rightarrow \infty$ is $|x| / 4$. Alternatively, using the root test, we have $\left|a_{n}\right|^{1 / n}=\frac{n^{1 / n}|x|}{4}$ whose limit as $n \rightarrow \infty$ is $|x| / 4$. Either way we see the series converges for $|x|<4$. If $x=4$ then the series is $\sum_{n=1}^{\infty}(-1)^{n+1} n$ which does not converge, and if $x=-4$ the series is $\sum_{n=1}^{\infty}(-n)$ which also does not converge. Therefore, the interval of convergence is $(-4,4)$.
(b) [2 points: 1 for correctly plugging in series, 1 point for answer]

We have $\lim _{x \rightarrow 0} \frac{f(x)-x / 4}{x^{2}}=\lim _{x \rightarrow 0} \frac{\left(x / 4-x^{2} / 8+3 x^{3} / 4^{3}-\ldots\right)-x / 4}{x^{2}}=\lim _{x \rightarrow 0} \frac{-x^{2} / 8+3 x^{3} / 4^{3}-\ldots}{x^{2}}=$ $\lim _{x \rightarrow 0}\left(-1 / 8+3 x / 4^{3}-\ldots\right)=-1 / 8$, where the last step is valid because the series for $f(x)$ converges to a continuous function at $x=0$.
(c) [2 points: 1 for correctly setting up integration, 1 point for answer]

Since the series converges absolutely for $0 \leq x \leq 1$, we may simply integrate term-byterm. Therefore, $\int_{0}^{1} f(x) d x=\int_{0}^{1}\left[\frac{1}{4} x-\frac{2}{4^{2}} x^{2}+\frac{3}{4^{3}} x^{3}-\frac{4}{4^{4}} x^{4}+-\cdots+(-1)^{n+1} \frac{n}{4^{n}} x^{n}+\cdots\right] d x$ $=\left.\left[\frac{1}{2 \cdot 4} x^{2}-\frac{2}{3 \cdot 4^{2}} x^{3}+\frac{3}{4 \cdot 4^{3}} x^{4}-\cdots+(-1)^{n+1} \frac{n}{(n+1) 4^{n}} x^{n+1}+\cdots\right]\right|_{x=0} ^{1}$ $=\frac{1}{2 \cdot 4}-\frac{2}{3 \cdot 4^{2}}+\frac{3}{4 \cdot 4^{3}}-\cdots+(-1)^{n+1} \frac{n}{(n+1) 4^{n}}+\cdots$.
(d) [3 points: 1 for answer, 1 for invoking alternating series test or Taylor estimate, 1 for establishing needed bound]
The given series for the integral is an alternating series whose terms are decreasing in magnitude and have limit zero. Therefore, by the alternating series test, the error in estimating the sum is at most the absolute value of the first omitted term. If we use the partial sum $\frac{1}{2 \cdot 4}-\frac{2}{3 \cdot 4^{2}}+\frac{3}{4 \cdot 4^{3}}-\frac{4}{5 \cdot 4^{4}}$, the next omitted term is $\frac{5}{6 \cdot 4^{5}}<1 / 1000$, so the given partial sum provides the desired estimate. We could also include more terms in the sum if desired. (One may also use Taylor's remainder theorem to estimate the error, but this is more difficult.)

