Lovely as a Tree Amplitude: Hidden Structures Underlie Feynman Diagrams

String theory, Fourier-conjugate spinors, and quantum chromodynamics are linked in ways that suggest new methods to compute probability amplitudes for high-energy particle interactions.

It isn't often that string theorists and particle physicists interested in phenomenology find themselves excited about the same body of work. But that's what happened earlier this year at the Kavli Institute for Theoretical Physics in Santa Barbara, California, during a three-month long workshop on collider physics. "It's the first time in living memory these two opposite wings of particle theory are collaborating on something real," observed workshop participant and Fermilab theorist Joseph Lykken.

The work that has inspired the common interest centers around Edward Witten of the Institute for Advanced Study in Princeton, New Jersey, and several of his colleagues. Though perhaps best known as a string theorist, Witten began his career studying the deep inelastic scattering of leptons on nucleons. Last December, he discovered a remarkable connection between a certain type of string theory and the weak-coupling regime of quantum chromodynamics (QCD) probed by high-energy accelerator experiments.[1] The connection suggests that a deep structure underlies the perturbative expansion used to analyze such experiments. In April, Witten, postdoc Freddy Cachazo, and graduate student Peter Svrcek exploited that structure to obtain greatly simplified expressions for leading perturbative terms.[2]

Helicity

More than 50 years ago, Richard Feynman showed how the terms that are summed in a perturbative calculation of a quantum probability amplitude can be represented by the simple diagrams that now bear his name. The terms of lowest order in $\hbar$ correspond to so-called tree diagrams. The name derives from the appearance of such diagrams—they comprise vertex points and lines. Vertices symbolize interactions, lines emerging from the diagram correspond to particles, and lines connecting pairs of vertices are called propagators. In higher-order perturbative diagrams, propagators form loops.

When CERN's Large Hadron Collider is up and running in a few years (see figure 1), it will smash protons against protons with 14 TeV center-of-mass energy. In some of those collisions, individual quarks will collide with each other at several TeV to produce jets—highly collimated sprays of hadrons that are the visible manifestation of individual high-energy quarks or gluons (the carriers of the strong force analogous to the photon) scattered at large angles. Figure 2 illustrates a generic Feynman diagram for a collision that yields four jets, a process commonly observed at Fermilab's 2 TeV Tevatron. Satisfactory recursive numerical algorithms exist for summing tree-level diagrams associated with jet production.

Witten and colleagues' recent theoretical contributions have shed light on the analytic structure of tree-level diagrams for which all particles are gluons. Symmetries relate such gluon diagrams to diagrams involving the highly energetic, effectively massless quarks that will be produced at the LHC.

In principle, writing down an analytic expression for the sum of tree-level diagrams with a specified number of gluons is straightforward; any standard field-theory text gives a recipe. In practice, applying the textbook recipe leads to extremely cumbersome expressions, because the natural, covariant rules for evaluating the Feynman diagrams force the use of redundant, inefficient variables.

The quantum amplitudes are functions of the four-momenta of incoming and outgoing gluons, but also depend on their spins. Those spins may be characterized by helicity—positive if spin and momentum are parallel, negative if they are antiparallel. In figure 2, plus and minus signs illustrate helicities. By convention, helicities in such diagrams are assigned as if all particles were outgoing. For ingoing particles you need to reverse the indicated helicity.

In a covariant expression for all-gluon Feynman diagrams, each

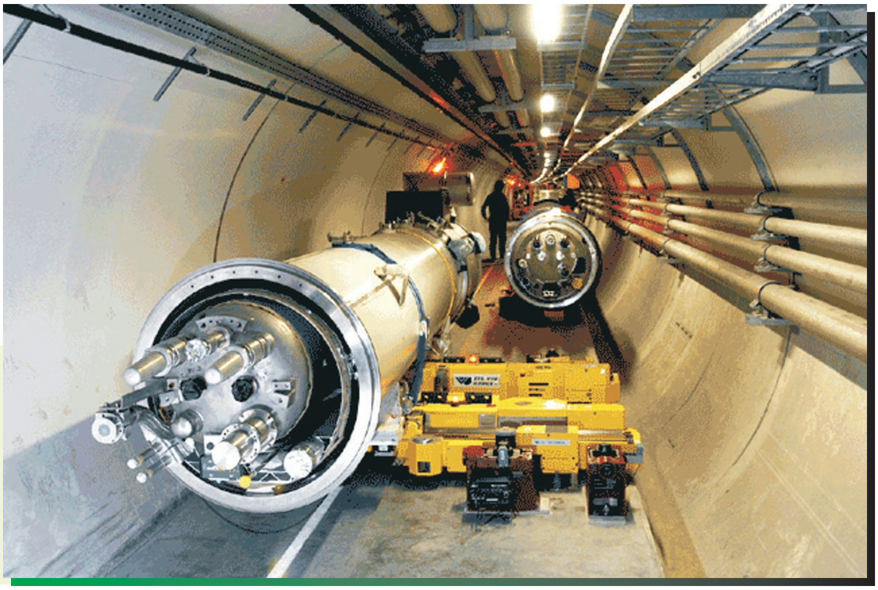


Figure 1. A Cryomagnet that will operate at CERN's Large Hadron Collider was positioned on its jacks during a test conducted last January. Scheduled to be producing physics results before the end of the decade, the LHC will be by far the world's highest-energy particle accelerator. (Courtesy of CERN.)

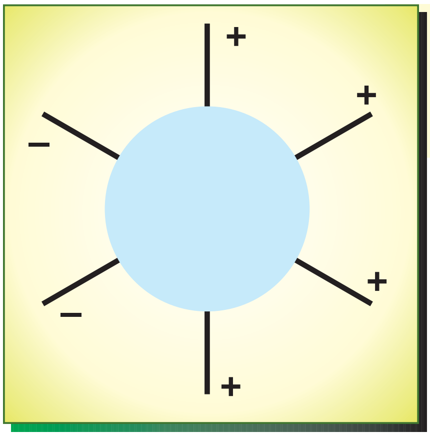


Figure 2. A generic Feynman diagram with six external lines symbolizes the production of four jets. Two of the lines represent the incoming colliding particles, and four represent the outgoing particles that generate the jets. In a tree-level diagram, the shaded region would contain a specific set of interaction vertices and internal straight lines called propagators. The plus and minus signs denote helicities.

gluon's helicity is described by an overdetermined four-vector. The symmetries of the theory guarantee that, when the smoke clears, the apparent extra degrees of freedom have no effect on the physics. But they do affect the complexity of the expression.

A different approach would be to classify the amplitudes in terms of the number of particles of each helicity and, once that's specified, try to determine the amplitudes as a function of momenta. In 1986, Stephen Parke and Tomasz Taylor proposed a compact formula for a subclass of tree diagrams, called maximal helicity-violating (MHV) diagrams. Their conjecture was proved two years later by Fritz Berends and Walter Giele. Witten's recent work with Cachazo and Svrcek showed how the Parke–Taylor formula could be generalized.

A process guaranteed to flip the helicities of all incoming particles would have the helicities in figure 2 be all of the same sign, conventionally taken as positive. However, the all-plus amplitudes vanish, as do amplitudes with just a single negative helicity. The MHV amplitudes have two minus signs and the rest positive.

As figure 3 shows, non-MHV amplitudes can be constructed by connecting MHV diagrams with one or more propagators. According to the standard Feynman rules, the internal propagators are "off-shell"—that is, their squared four-momenta do not vanish. And there's the rub: The Parke–Taylor formula, appropriate when all particles are massless, does not apply. Witten and his two colleagues showed how to generalize the formula to apply to the off-shell MHV diagrams that are sewn together with propagators to yield non-MHV amplitudes.

The construction of the non-MHV amplitudes (called the CSW construction after its inventors) was not proved from first-principles. But UCLA's Zvi Bern, one of the organizers of the Kavli workshop, is convinced that it is correct. He says that, in a number of nontrivial cases, the CSW construction has been checked numerically to 15 or more decimal places. Moreover, the construction satisfies certain necessary analytic "factorization" properties that strongly suggest its veracity. Indeed, before Witten, working with his colleagues, wrote down the explicit CSW construction, he had already discovered how the desired factorization properties could arise in non-MHV amplitudes. Witten's earlier work suggested deep structures behind Feynman diagrams with massless particles and pointed to profound connections between string theory and perturbative QCD. Spinor coordinates helped make the connection.

Let's do the twistor

Two spin-1/2 particles can combine to give a spin-1 object. Conversely, a spin-1 particle can be viewed as a combination of two spin-1/2 objects; in the language of rotation-group theory, one says that a vector can be expressed in terms of two spinors. Lorentz symmetry allows four-momenta of massless particles to be written as a product of two Lorentz-group spinors. The spinor decomposition affords a fresh view of tree amplitudes as functions of spinor momenta instead of four-vector momenta.

Witten considered tree-level amplitudes expressed in the spinor language and explored their properties under so-called conformal symmetry transformations. The symmetry operations could be expressed, by multiplication and differentiation, with spinor coordinates, but the expressions were awkward in those coordinates. Witten noticed that when amplitudes are Fourier transformed with respect to one of the two spinor coordinates—an operation that exchanges the roles of multiplication and differentiation for that coordinate—then the conformal symmetry operations have more satisfying, transparent expressions. That Fourier transform brought Witten to twistor space, the space whose coordinates are those of a spinor coupled with a Fourier-conjugate spinor.

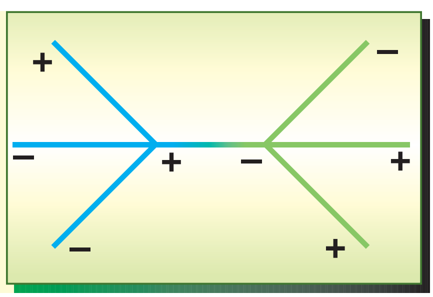
Twistor theory was invented by Roger Penrose in the 1970s, but was applied by only a passionate few through the remainder of the 20th century. Penrose's aim was to offer a new perspective on quantum mechanics and general relativity. He used mathematical techniques that were then unknown to physicists. The later application of those techniques to string theory gave insights that Witten exploited in his twistor-space investigations. That recent work, Witten says, may help bring twistor theory into the mainstream.

When expressed in terms of twistor coordinates, tree-level amplitudes have a number of remarkable features. For example, suppose one asks if there is a special property relating the momenta of nonvanishing tree amplitudes. When the momenta are expressed as four-vectors, nothing is apparent. Witten discovered that when the momenta of nonvanishing MHV amplitudes are expressed in twistor-space coordinates, they all lie on a straight line. Momenta for nonzero, non-MHV amplitudes with three minus signs lie on curves described by quadratic equations; momenta for amplitudes with four minus signs lie on curves defined by cubic equations, and so forth.

A string duality

Inspired by the remarkable properties of the twistor-space amplitudes, Witten looked for specific calculations in the context of a string theory that would reproduce the results conventionally calculated with QCD tree amplitudes. Dualities relating string theories and gauge theories had been

Figure 3. Sewing together two maximal helicity-violating diagrams with a propagator yields a non-MHV tree diagram with three negative helicities. By iterating the procedure, one can construct arbitrary non-MHV tree diagrams.



developed before, but they related strings to the strong-coupling regime in which quarks and gluons are confined (see PHYSICS TODAY, August 1998, page 20).

Witten was seeking a duality that would relate strings to perturbative QCD. He conjectured that the correct string theory was a so-called topological B-model, in which a string's two-dimensional spacetime world sheet is mapped into twistor space. The "topological" in the model's name refers to a mathematical device that prohibits the string from having massive excited states; by contrast, conventional string models give an infinite number of massive states. Topology enters into Witten's construction in a second way: The so-called instanton string configurations that reproduce QCD amplitudes wrap around topological structures in twistor space. That wrapping is analogous to the way in which a piece of sticky tape can be attached to the surface of a bagel, thread the interior hole, and connect to itself.

Witten's string–QCD duality, like the CSW construction, is a conjecture. But even before the CSW result was available, Radu Roiban (University of California, Santa Barbara), working with the Kavli Institute's Marcus Spradlin and Anastasia Volovich,[3] used the duality to obtain five-gluon amplitudes in which two of the external helicities were positive and three were negative.

How were Roiban and colleagues able to make progress without the CSW result? They chose a diagram that was an MHV diagram with all the helicity signs reversed. Borrowing terminology from Penrose, Witten called such diagrams "googly," from the game of cricket. A googly is a slow-pitched ball whose artfully disguised spin is opposite to what the batsman expects based on the bowler's action. Had twistor theory been developed in North America rather than England, we might be speaking of "screwball amplitudes." Googly amplitudes are simply the complex conjugates of the corresponding MHV trees.

The string-theory calculation implemented by Roiban and company to obtain the five-gluon googly amplitude was somewhat different from that originally conjectured by Witten. Understanding just which stringy algorithms yield tree amplitudes and what are the relationships between those algorithms is a work in progress.

The twistor-space results and new string duality conjectured by Witten have led to simplified algorithms for calculating tree diagrams. But to test QCD precisely, one needs to go beyond tree level and calculate Feynman diagrams with loops. A fruitful approach to one-loop diagrams, one that meshes well with the twistor approach, has been carried furthest by Bern, Lance Dixon (SLAC), and David Kosower (Atomic Energy Comission, Saclay, France). They have determined the analytic properties such diagrams must possess and have mathematically constructed amplitudes with the needed properties. The CSW construction may allow such techniques to be expanded to a new regime of loop diagrams. A number of groups are currently exploring whether the twistor-space methods that yielded insight into trees might illuminate loops as well.

"There's been a story with many chapters of applying stringlike methods to gauge theories," says Witten. "My work was a little bit closer to what's relevant to current collider physics. And that makes me happy."

Steven K. Blau

References

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A Dark-Horse Entry in the Race for an Excitonic Condensate

Evidence for a superfluid made of electron–hole pairs comes, surprisingly, from quantum Hall systems.

If electron pairs can condense into a macroscopic superconducting state, can't electron–hole pairs—excitons—similarly condense to form a neutral superfluid? Such was the speculation of several theorists in the 1960s.[1] Experimenters have been searching ever since for excitonic condensates. They have focused the search on the electron–hole pairs created by shining light onto a semiconductor. Such studies have turned up some intriguing behavior, but still no conclusive proof of a condensate.

The long-sought evidence for an excitonic condensate has now surfaced in a different and unexpected quarter: a quantum Hall bilayer. The bilayer is composed of two slabs of doped semiconductors separated by a thin insulating region. Each slab functions as a two-dimensional electron (or hole) gas. A strong magnetic field is applied perpendicular to the layers, and the charges move in quantized orbits about the field lines.

Because its two layers are doped with the same charge, a quantum Hall bilayer seems an unlikely place to find excitons. However, a remarkable coherence develops between the charges in the two layers when the magnetic field and the layer separation take on just the right values. In negatively doped bilayers, conduction-band electrons in one layer become acutely aware of those in the other, and they coordinate their spatial arrangement, as seen in figure 1. Each electron lines up opposite a vacancy in the conduction band—that is, a hole. These opposite charges bind as excitons that

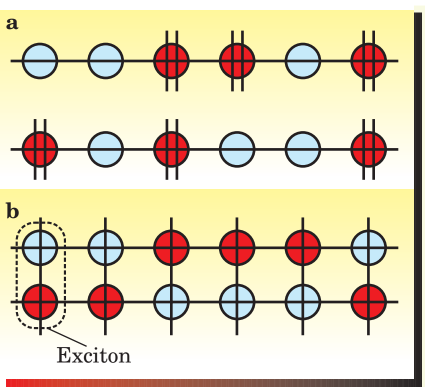

Figure 1. Excitons form when parallel layers of two-dimensional electron gases are close enough and when a perpendicular magnetic field provides two flux quanta (vertical black lines) for each electron (red circles). **(a)** At large separations, electrons act independently of one another. **(b)** At small separations, the electrons arrange themselves as if they were in the same layer, and each is associated with only one flux line. The electrons bind to the vacancies, or holes, (blue circles) in the opposite layers to form excitons.