Left and right convergence of bounded degree graphs

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Graph Convergence

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Connected? Bipartite? Largest cut? Number of triangles?

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We will discuss sequences where the maximum degree of each G_n is bounded by constant D.

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For every r, gives a distribution on a finite set of rooted connected graphs. Does this distribution converge?

Graph homomorphism counts

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$ind(G, H)$ number of such maps where the image of $V(G)$ is an induced subgraph isomorphic to G.

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$$t(G, H) = \frac{\hom(G, H)}{|V(H)|^{|V(G)|}}$$

Left Convergence alternative definition

Left Convergence

Given a sequence of graphs (G_n) with degrees bounded by D, we it is *left convergent* if for any connected graph F, the following limit exists:

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Given a sequence of graphs (G_n) with degrees bounded by D, we it is *left convergent* if for any connected graph F, the following limit exists:

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We can replace ind with inj or hom.

For example:

$$\mathrm{inj}(F, G) = \sum_{F' \supseteq F} \mathrm{ind}(F', G)$$

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For example, a random sequence of d-regular graphs, interlaced with a random sequence of d-regular bipartite graphs is left convergent.

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Given a sequence of graphs (G_n) with degrees bounded by D, we say that it is *right convergent with soft-core constraints* if for any weighted graph H with positive weights (including on loops), the following limit exists:

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Equivalently, can look at

$$\lim_{n\to\infty} \frac{\log t(G_n, H)}{|V(G_n)|} = \lim_{n\to\infty} \frac{\log \hom(G_n, H)}{|V(G_n)|} - \log |V(H)|$$

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Theorem (Borgs, Chayes, Gamarnik)

Given a sequence G_n of bounded degree graphs, if it is soft-core right convergent, then we can remove $o(|V(G_n)|)$ edges from G_n to make it hard-core right convergent.

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Right convergence is strictly stronger than left convergence.

For example, right convergence implies convergence of maxcut.

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(L.)

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Proof only considers H with positive weights, gives a direct proof that soft-core right convergence implies left convergence.

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$X(G, k) \in [0, D]^{k+\binom{k+1}{2}}$, in a bounded region.

Alternative definition of right convergence

Observation

Let $\lambda \in \mathbb{R}^{k+\binom{k+1}{2}}$. H_λ graph on $[k]$ with vertex weights e^{λ_i}, edge weights $e^{\lambda_{ij}}$. Then

$$\mathbb{E}[e^{\langle \lambda, Y(G,k) \rangle}] = t(G, H_\lambda)$$

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Definition

Define the normalized cumulant generating function of Y:

$$f_{G,k}(\lambda) = \frac{\log \mathbb{E}[e^{\langle \lambda, Y(G,k) \rangle}]}{|V(G)|} = \frac{\log \mathbb{E}[e^{|V(G)| \langle \lambda, X(G,k) \rangle}]}{|V(G)|}$$

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Soft-core right convergence is equivalent to pointwise convergence of $f_{G_n,k}$ for all k.

Proof outline

Proposition

Given a sequence G_n of graphs with degree bounded by D, left convergence is equivalent to convergence of all partial derivatives at $\lambda = 0$ of each $f_{G_n,k}$ as n goes to infinity.

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If G is a graph with degrees bounded by D, then for any λ_0 with $\|\lambda_0\|_\infty \leq 1$, if we let $g(z) = f_{G,k}(z\lambda_0)$, then

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Implies that Taylor series converges uniformly for λ with $\|\lambda\|_\infty < (4eD)^{-1}$.

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Joint Cumulant

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If $Z_1, Z_2, ..., Z_r$ can be partitioned into two groups such that the two groups are independent, then $\kappa(Z_1, Z_2, ..., Z_r) = 0$.

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So we can also write

$$\kappa_r(Z) = g^{(r)}(0)$$

where

$$g(t) = \log \mathbb{E}\left[e^{tZ}\right]$$

Partial derivative of $|V(G)|f_{G,k}(\lambda)$ by $\lambda_{i_1,j_1}, \lambda_{i_2,j_2}, ..., \lambda_{i_l,j_l}$ is equal to

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Let H be the graph formed by $(i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l)$. This allows us to write

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There is also a formula

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Using this, can show that for every l, we can take k large enough to make this invertible.

Other notions of convergence

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Sequence of n C_4's and n C_6's interlaced is partition convergent.

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Large Deviations Convergence (Borgs, Chayes, Gamarnik):

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Implies both right and partition convergence, strictly stronger.

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If we color vertices of G_n with k colors as well, can define, for each $F \in \mathcal{F}_k$, values $\hom(F, G_n)$, $\text{inj}(F, G_n)$, $\text{ind}(F, G_n)$ analogously.

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Colored-neighborhood convergence: any k, any $\mathcal{F}'$, sets S_n are Hausdorff-convergent.

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Does LD-convergence imply colored-neighborhood convergence?

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Let G_n be a sequence of random d-regular graphs.

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2. WHP G_n is colored-neighborhood convergent.

Very difficult problem, lots of implications